B. Sc. Part I (HONS) 1st Paper

TRIGONOMETRY

Summation of series

Important Formulae

$$\sin\alpha + \sin(\alpha+\beta) + \cdots + \sin\{\alpha+(n-1)\beta\}$$

$$= \frac{\sin\left\{\alpha + \frac{(n-1)\beta}{2}\right\}}{\sin\frac{\beta}{2}} \cdot \sin\frac{n\beta}{2}$$

$$\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos\{\alpha+(n-1)\beta\}$$

$$= \frac{\cos\left\{\alpha + \frac{(n-1)\beta}{2}\right\}}{\sin\frac{\beta}{2}} \cdot \sin\frac{n\beta}{2}.$$

Sums

1. Sum the following series to n-terms and to infinity where $a < 1$.

$1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots$

Soln

Let $C = 1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \cdots + a^{n-1}\cos(n-1)\theta$

and $S = a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \cdots + a^{n-1}\sin(n-1)\theta$

$\therefore C + iS$

$= 1 + a(\cos\theta + i\sin\theta) + a^2(\cos 2\theta + i\sin 2\theta) + a^3(\cos 3\theta + i\sin 3\theta) + \cdots + a^{n-1}[\cos(n-1)\theta + i\sin(n-1)\theta]$

$\Rightarrow C + iS = 1 + ae^{i\theta} + a^2e^{2i\theta} + a^3e^{3i\theta} + \cdots + a^{n-1}e^{(n-1)i\theta}$

The series of RHS is in G.P.

Here, the common ratio $= ae^{i\theta}$

$$\text{Sum of GP (of } n \text{ terms)} = \frac{a(1-r^n)}{1-r}, \ 1>r$$

$$\Rightarrow C + iS = \frac{1 \cdot \left[1-(ae^{i\theta})^n\right]}{1-ae^{i\theta}} = \frac{1-a^n e^{in\theta}}{1-ae^{i\theta}}$$

$$\Rightarrow C+iS = \frac{1-a^n e^{in\theta}}{1-ae^{i\theta}} \times \frac{1-ae^{-i\theta}}{1-ae^{-i\theta}}$$

$$= \frac{1+a^{n+1}e^{i(n-1)\theta} - a^n e^{in\theta} - ae^{-i\theta}}{1-a(e^{i\theta}+e^{-i\theta})+a^2}$$

$$\Rightarrow C+iS = \frac{1-a^n(\cos n\theta + i\sin n\theta) - a(\cos\theta - i\sin\theta) + a^{n+1}\left[\cos(n-1)\theta + i\sin(n-1)\theta\right]}{1-2a\cos\theta+a^2}$$

Equating real parts, we get

$$C = \frac{1-a^n\cos n\theta - a\cos\theta + a^{n+1}\cos(n-1)\theta}{1-2a\cos\theta+a^2}$$

This is the sum of given series upto n-terms.

Now, the sum of the given series up to infinity

$$= \underset{n \to \infty}{Lt} \frac{1 - a^{n} \cos n\theta - a\cos\theta + a^{n+1}\cos(n-1)\theta}{1 - 2a\cos\theta + a^{2}}$$

Given that $a < 1 \Rightarrow \underset{n \to \infty}{Lt} a^{n} \to 0$

and $\underset{n \to \infty}{Lt} a^{n+1} \to 0$.

Hence, the sum of series up to infinity

$$= \frac{1 - 0 - a\cos\theta + 0}{1 - 2a\cos\theta + a^{2}} = \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^{2}}$$